On computing the pressure by the p version of the finite element method for Stokes problem

Soren Jensen[1]
Department of Mathematics and Statistics
University of Maryland, Baltimore County
Baltimore, Maryland 21228

February 15, 1990
Keywords: Hydrostatic pressure, Stokes problem, p version, stability.
Subject Classifications AMS(MOS): 65N30, 65N15, 76D07; CR:5.17.

[1]partially supported by ONR grants N00014-87-K-0427 and N00014-90-J-1238

Abstract

This paper introduces and analyzes two ways of extracting the hydrostatic pressure when solving Stokes problem using the *p* version of the finite element method. When one uses a local H^1 projection, we show that optimal rates of convergence for the pressure approximation is achieved. When the pressure is not in H^1, or the value of the pressure is only needed at a few points, one may extract the pressure pointwise using e.g. a single layer potential recovery. Negative norm and interior estimates for the Stokes velocity are derived within the framework of the *p* version of the F.E.M.

1. Introduction.

There are cases in which a continuum is subjected to an incompressibility constraint, such as a divergence constraint. We will investigate some problems dealing with stability of a class of numerical discretizations of such problems. As an important example we consider Stokes equations in two dimensions:

$$\begin{aligned} -\Delta \mathbf{U} + \nabla P &= \mathbf{F} \quad \text{in} \quad \Omega \subseteq \mathbf{R}^2 \\ \nabla \cdot \mathbf{U} &= 0 \quad \text{in} \quad \Omega \end{aligned} \tag{1.1}$$

with appropriate boundary conditions on $\partial\Omega$. In the standard weak formulation,

$$\begin{aligned} &\text{Find } \mathbf{U} \in V \subseteq [H^1(\Omega)]^2 \quad \text{and} \quad P \in W \subseteq L^2(\Omega) \text{ such that} \\ a(\mathbf{U},\mathbf{v}) + b(\mathbf{v},P) &= (\mathbf{F},\mathbf{v}) \qquad \forall \mathbf{v} \in V \\ b(\mathbf{U},q) &= 0 \qquad \forall q \in W \end{aligned} \tag{1.2}$$

where the bilinear forms a and b are given by

$$\begin{aligned} a(\mathbf{U},\mathbf{v}) &= 2\int_\Omega \sum_{i,j} \epsilon_{ij}(\mathbf{U})\epsilon_{ij}(\mathbf{v})d\mathbf{x} \\ b(\mathbf{v},P) &= -\int_\Omega \nabla \cdot \mathbf{v} P d\mathbf{x} \end{aligned} \tag{1.3}$$

and $(\mathbf{F},\mathbf{v})$ denotes the usual $[L^2(\Omega)]^2$ inner product. $\epsilon_{ij}(\mathbf{v})$ is the symmetric part of the deformation gradient $\frac{1}{2}(\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j})$. The pair of spaces (V,W) depends on the boundary conditons. For no-slip boundary conditions,

$$(V,W) = ([H_0^1(\Omega)]^2, \{q \in L^2(\Omega) : \int_\Omega q d\mathbf{x} = 0\})$$

and for stress-free boundary conditions,

$$(V,W) = (\{\mathbf{v} \in H^1(\Omega) : \epsilon_{ij}(\mathbf{v}) = 0\}^\perp, L^2(\Omega)).$$

We discretize by choosing finite dimensional spaces $V_N \subseteq V, W_N \subseteq W$, such that we are to

$$\begin{aligned} &\text{Find } \mathbf{U}_N \in V_N \quad \text{and} \quad P_N \in W_N \text{ such that} \\ a(\mathbf{U}_N,\mathbf{v}) + b(\mathbf{v},P_N) &= (\mathbf{F},\mathbf{v}) \qquad \forall \mathbf{v} \in V_N \\ b(\mathbf{U_N},q) &= 0 \qquad \forall q \in W_N \end{aligned} \tag{1.4}$$

The main problem is whether it is possible to select a sequence of pairs (V_N, W_N) that is stable and that has good approximation properties. The Babuška - Brezzi condition

$$\min_{q\in W_N\setminus\{0\}} \max_{\mathbf{v}\in V_N\setminus\{0\}} \frac{\int_\Omega \nabla \cdot \mathbf{v}\, q\, d\mathbf{x}}{\| \mathbf{v} \|_{H^1} \| q \|_{L_2}} \geq c > 0 \tag{1.5}$$

with c independent of N, is sufficient to guarantee stability in the sense that the discretization errors are within a uniform constant of the distances from $(\mathbf{U}, P)$ to (V_N, W_N) (cf. [4],[8]).

We shall concern ourselves with methods using high degree polynomials to approximate the solution to the Stokes equations. Among possible choices: spectral methods (cf. [11]) and the p-version of the finite element method, we shall deal with the latter. Accuracy is improved by increasing the polynomial degree p in the (usually small number of) elements in the mesh, see [5].

We saw in [16] that the pair $([Q^p]^2, \nabla \cdot Q^p)$ was unstable, since the right inverse $(\nabla\cdot)_p^{-1} : W_N \rightarrow V_N$ had a norm in $B(L^2, H^1)$ that was growing linearly with p. When $W_N = \nabla \cdot V_N$, it is well known that at uniform bound on $(\nabla\cdot)^{-1}$ in $B(L^2, H^1)$ is equivalent to the Babuška - Brezzi condition (cf. [23]). Similar results hold for other pairs in which the degrees of the polynomials in the (V_N, W_N) pair is one less in W_N than in V_N, see [16]. That the inf-sup constant tends to 0 like $\frac{1}{p}$ as $p \rightarrow \infty$ does not mean automatic suboptimal covergence for the velocity as was explained in [16], e.g. But the pressure convergence rate can be slowed down by 1 as was seen in an example there, and following another example does not necessarily have to, see [16]. We will try to recover the pressure through some additional effort - solving a Poisson problem - at optimal rate.

From (1.1) we see that P satisfies

$$\begin{array}{rcl} \Delta P & = & \nabla \cdot \mathbf{F} \quad \text{in} \quad \Omega \\ \frac{\partial P}{\partial n} & = & (\mathbf{F} + \Delta \mathbf{U}) \cdot \mathbf{n} \quad \text{on} \quad \partial\Omega \end{array} \tag{1.6}$$

and $\int_\Omega P dx = 0$. $\mathbf{n}$ denotes the unit outward normal defined almost everywhere. (1.6) is a Poisson problem for which the accuracy of any numerical procedure will depend on the accuracy of approximation of $\mathbf{U}$.

2. Stability and velocity convergence.

A lack of stability (deterioration of the inf-sup constant) does not imply suboptimal convergence of the velocities. Let

$$Z_N = \{\mathbf{v} \in V_N : \mathbf{v} = 0 \text{ on } \partial\Omega, \nabla \cdot \mathbf{v} = 0 \text{ in } \Omega\} \tag{2.1}$$

where we have taken no-slip boundary conditions. Let $W_N \supseteq \nabla \cdot V_N$. Then $\mathbf{U}_N$ is an elliptic projection of $\mathbf{U}$ onto Z_N such that

$$\| \mathbf{U} - \mathbf{U}_N \|_{1,\Omega} \leq \min_{\mathbf{v} \in Z_N} \| \mathbf{U} - \mathbf{v} \|_{1,\Omega} \tag{2.2}$$

where $\| \cdot \|_{s,\Omega} = \| \cdot \|_{H^s(\Omega)}$ the standard Sobolev norm, see [1]. Therefore, there exist stream functions $\varphi, \varphi_N \in H_0^2(\Omega)$ such that

$$\mathbf{U} = \nabla \times \varphi, \qquad \mathbf{U}_N = \nabla \times \varphi_N \tag{2.3}$$

where $\nabla \times \varphi = (-\frac{\partial \varphi}{\partial y}, \frac{\partial \varphi}{\partial x})$. Let $R = (-1,1)^2$ and

$$Q^p = \text{span}\{x^m y^n : 0 \leq m, n \leq p\} \tag{2.4}$$

be the spaces of polynomials of separate degree less than or equal to p. We now have for one square element.

Proposition 2.1 *If $V_N = [Q^p \cap H_0^1(R)]^2$ and $W_N = \nabla \cdot V_N$, then for $p \geq 4$, $\mathbf{U}_N$ is an elliptic projection onto Z_N with*

$$\| \mathbf{U} - \mathbf{U}_N \|_{1,R} \leq C p^{-M} \| \mathbf{U} \|_{M+1,R} \tag{2.5}$$

Proof: Let $Y_N = ((\nabla \times)^{-1}(Z_N)) \cap H_0^2(R)$. Then

$$\begin{aligned}
\min_{\mathbf{v} \in Z_N} \| \mathbf{U} - \mathbf{v} \|_{1,R} &\leq \min_{\psi \in Y_N} \| \nabla \times (\varphi - \psi) \|_{1,R} \\
&\leq \min_{\psi \in Y_N} \| \varphi - \psi \|_{2,R} \\
&\leq C p^{-M} \| \varphi \|_{M+2,R} \\
&\leq C p^{-M} \| \mathbf{U} \|_{M+1,R}
\end{aligned}$$

which combined with (2.2) and the observation that for $p \geq 4, Y_N = Q^p \cap H_0^2 = (1-x^2)^2(1-y^2)^2 Q^{p-4} \neq \emptyset$ and has optimal approximation properties, using an interpolation argument due to Babuška: Optimal rates of approximation were established in [25] for $M > \frac{3}{2}$ and are clear at $M = 0$, upon which it follows through interpolation by the K-method for intermediate values. The H_0^2 trace constraints interpolate as expected, as shown in [15].

□

Remark 2.2 When $\mathbf{F}$ is $\mathbf{0}$ or sufficiently smooth, the rate appearing in (2.5) is not optimal for typical corner singularities that appear in the solution of elliptic boundary value problems on polygonal domains. It follows from the analysis in [6], that if $\mathbf{U}$ is of the form $r^\alpha |\log r|^q \phi(\theta)$ – expressed in polar coordinates at the corner – and ϕ is smooth, then

$$\min_{\mathbf{v} \in V_N} \| \mathbf{U} - \mathbf{v} \|_1 \leq C_\epsilon p^{-2\alpha+\epsilon}$$

which is near twice the rate of (2.5) as $\mathbf{U} \in H^s$ for $s < \alpha + 1$. Let us briefly recap approximation theoretical results for the stream function φ. Let $\tilde{\varphi} = r^\gamma |\log r|^q \phi(\theta)$

($\gamma = \alpha + 1$). Combining the results in [12], Chapter 7, and [9], one sees that for corner angles $\omega \stackrel{<}{\sim} .8128\pi$, the first non Gaussian integer pole of a certain transcendental equation, which determines the strength γ, is double and $q = 1$. In [25] it was shown that then the following approximation result holds

$$\min_{\psi \in Y_N} \| \varphi - \psi \|_2 \leq C |\log p| p^{-2(\gamma-1)}. \tag{2.6}$$

with C independent of p. By the above interpolation argument one can get $|\log p|^\theta$ for any $0 < \theta < 1$ and C independent of θ. This indicates that the log factor is not important in (2.6). The singular form of the solution

$$\tilde{\varphi} = r^\gamma |\log r|^q \phi(\theta) \tag{2.7}$$

can also stem from the specification of Dirichlet data for $\mathbf{U}$ on the boundary. We shall mention an example of this later and note now that in such a case, the $|\log p|$ factor is not present iff it is not present in the φ satisfying the boundary conditions and this singularity is stronger than that inherited from the geometry.

In view of the previous remark, let us introduce a rate of approximation function $\Re$ which expresses the hitherto known results, cf. thm. 6.1 in [25]

$$\min_{\psi \in Y_N} \| \varphi - \psi \|_2 \leq C \Re(p, t, \rho) \tag{2.8}$$

where ρ is a regularity index:

$$\rho(\varphi) = \begin{cases} (0,0,s) & \begin{cases} \text{when } \varphi \in H^s \setminus H^{s+\epsilon},\ \forall \epsilon > 0 \\ \text{is assumed known only,} \end{cases} \\ (\gamma, q, 1+\gamma-\epsilon) & \begin{cases} \text{when } \varphi \in H^{1+\gamma-\epsilon} \setminus H^{1+\gamma},\ \forall \epsilon > 0 \text{ and} \\ \text{has known dominating singularity:} \\ \tilde{\varphi} = r^\gamma |\log r|^q \phi(\theta) \text{ in local polar coordinates} \\ \text{at the boundary and } \varphi - \sum \tilde{\varphi} \in H^s,\ s > 2\gamma \end{cases} \end{cases} \tag{2.9}$$

so that we may define

$$\Re = \begin{cases} p^{(2-t)} \| \varphi \|_t & \text{when } \rho(\varphi) = (0,0,s),\ t \leq s \\ |logp|^q p^{-2(\gamma-1)} R(\varphi) & \text{when } \rho(\varphi) = (\gamma, q, 1+\gamma-\epsilon),\ \epsilon > 0 \end{cases} \tag{2.10}$$

$R(\varphi)$ involves $\|\varphi - \sum \tilde{\varphi}\|_s$ and the l_1 norm of the vector of coefficients of $\tilde{\varphi}$.

Statements similar to those in prop. 2.1 can be made for spaces of polynomials of total degree $\leq p$ and those used in PROBE [28] for $p \geq 7$. These are defined as

follows

$$
\begin{aligned}
P^p \quad &= \text{span}\{\ x^m y^n : 0 \le m, n \le p\} \\
S^p \quad &= \text{span}\{\ x^m y^n : 0 \le m, n;\ m + n \le p \\
&\qquad\qquad \text{or } (m = p \wedge n = 1) \text{ or } (m = 1 \wedge n = p)\} \\
&= P^p \otimes \{\, x^p y, x y^p \},
\end{aligned}
\tag{2.11}
$$

respectively. The latter is convenient when approximating Dirichlet boundary conditions. If one relaxes the no-slip boundary conditions along part of the boundary, the threshold values decrease accordingly.

$([P_0^p]^2, P^{p-1})$ – or any other pair with a similar degree selection – is a natural choice in the sense that it reflects the approximability of the velocity - pressure pair given by their regularity and that it was proven h-stable for $p \ge 4$ when excluding exceptional meshes, see [23]. The stability does not seem to be restored by merely restricting the pressures to lie in P^{p-2}. [22] reports the behavior of the inf-sup constant as $p^{-1/4}$ for large p. In fig. 1 we give the smallest positive singular value of the discrete Stokes matrix for $([P_0^p]^2, P^{p-3})$. Again $p^{-1/4}$ seems to be the asymptotic behaviour.

It is possible to stabilize at a cost. In [13] and [14] , etc., it has been proposed to add weighted least squares terms to the energy. The problem possesses coercivity and is only in the very limit of saddle point nature. However, it seems that the least squares terms added require that $\mathbf{U} \in [H^2]^2$, which is more regularity than we would like to require a priori. In fact, the examples computed in [16] indicate that H^2 regularity of $\mathbf{U}$ was sufficient for the pressure to be approximated optimally by the mixed p F.E.M. We will give a brief explanation of this later. One way to get a stable (V_N, W_N) pair is derived in the following example.

Example 2.3 A natural idea for proving a uniform lower bound on the discrete inf-sup constant is to attempt to inherit the continuous bound by constructing the maximal right inverse of the Div operator. This general avenue was thoroughly explored in [30], [23] and references listed there. In the context of the p version we refer to [16]. We will mimic the construction in the nondiscrete case where uniform bounds exist also for nonsmooth polygonal domains under Dirichlet boundary conditions, [2].

Let $R = (-1,1)^2$, $\Phi_N \subset P^{p+1} \cap H_0^1(R)$ and $W_N = \Delta \Phi_N$. Clearly

$$
\begin{tikzcd}
\Phi_N \arrow[rr, leftrightarrow, "\Delta", "\text{bijection}"'] \arrow[dr, "\nabla"'] & & W_N \\
& V_N \arrow[ur, "\nabla\cdot"'] &
\end{tikzcd}
$$

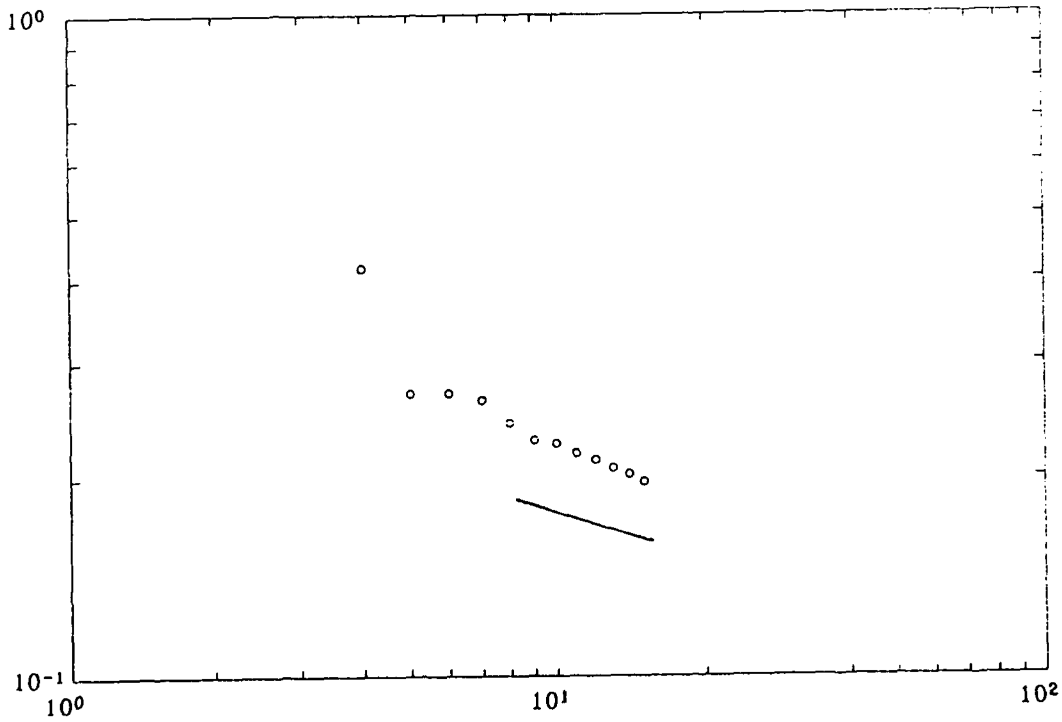

Figure 1: Discrete inf-sup constant as a function of p for $([P_0^p]^2, P^{p-3})$. Slope of line is $-\frac{1}{4}$.

with

$$\begin{aligned} \| (\nabla\cdot)^{-1} q \|_1 &= \| (\nabla\cdot)^{-1}\Delta\Delta^{-1} q \|_1 \\ &\leq \| \nabla(\Delta^{-1} q) \|_1 \\ &\leq \| \Delta^{-1} q \|_2 \\ &\leq C \| q \|_0 \quad \forall q \in L^2(R), \end{aligned}$$

where C depends only on R, not on q, giving a uniform bound on the operator norm of Div^{-1}. We may therefore select

$$\begin{aligned} W_N &= \Delta(P_0^{p+1}) \text{ and} \\ V_N &= \nabla(P_0^{p+1}) \end{aligned}$$

with no regard to boundary conditions and under no-slip boundary conditions, we instead set

$$\begin{aligned} W_N &= \Delta([N_1^{(0)}]^2 P^{p-7}) \text{ and} \\ V_N &= \nabla([N_1^{(0)}]^2 P^{p-7}) \end{aligned}$$

where $N_1^{(0)} = (1-x^2)(1-y^2)$ is the first internal shape function. As Div^{-1} was uniformly bounded in $B(L^2, H^1)$, the pair $(\nabla\Phi_N, \Delta\Phi_N)$ is p-stable, where $\Phi_N = [N_1^{(0)}]^2 P^{p-7}$.

$\square$

Both choices seem unnatural, however, from the regularity point of view as well as from the stand-point that you would like to be able to combine the p and the h versions.

3. Negative norm estimates

For future purposes we shall derive error estimates in negative norms for the velocities. As seen in section 2, it is possible to characterize the velocity convergence by that of stream functions. $\mathbf{U}_N$ is an elliptic projection of $\mathbf{U}$ onto Z_N so that

$$\| \mathbf{U} - \mathbf{U}_N \|_{H^s(\Omega)} = \| \nabla \times (\varphi - \varphi_N) \|_{H^s(\Omega)} \leq \| \varphi - \varphi_N \|_{H^{s+1}(\Omega)}$$

with

$$\begin{aligned} a(\nabla \times (\varphi - \varphi_N), \nabla \times \chi) &= 0, \quad \forall \chi \in Y_N = H_0^2(\Omega) \cap (\nabla \times)^{-1} Z_N \\ \Downarrow \qquad c(\varphi - \varphi_N, \chi) &= 0 \end{aligned}$$

where

$$c(u,v) = \int_{\Omega} \Delta u \Delta v dx \qquad \forall u,v \in H_0^2(\Omega)$$

(if stress-free b.c., get $\int_\Omega (u_{xx}v_{xx} + u_{yy}v_{yy} + 2u_{xy}v_{xy})dx$). Let $e = \varphi - \varphi_N$. We need to bound $\| e \|_{-k}$ for some nonnegative inteqer k.

Proposition 3.1 *Let $\Omega = (-1,1)^2, k \in \mathbf{N}$ and e be the difference between the exact (φ) and the discrete (φ_N) stream functions. Then*

$$\| e \|_{-k,\Omega} \leq C\Re(p,t,\rho(\varphi))p^{-k-2} \tag{3.1}$$

where $\Re$ was defined in (2.10).

Proof:

$$\| e \|_{-k,\Omega} = \sup_{v \in C_0^\infty(\Omega)\backslash\{0\}} \frac{(e,v)}{\| v \|_{k,\Omega}}$$

Now we define ψ to be the solution of

$$\begin{aligned} \Delta^2\psi &= v \qquad \text{in } \Omega \\ \psi &= \frac{\partial\psi}{\partial n} = 0 \text{ on } \partial\Omega \end{aligned} \tag{3.2}$$

$v \in C_0^\infty(\Omega) \Rightarrow \psi \in C^\infty(\overline{\Omega})$ and $\| \psi \|_{s,\Omega} \leq C \| v \|_{s-4,\Omega}$ as we will discuss in the remarks following the proof.

$$(e,v) = (e,\Delta^2\psi) = c(e,\psi) = c(e,\psi-\eta) \quad \forall \eta \in Y_N$$

such that

$$\begin{aligned} |(e,v)| &\leq \| e \|_{2,\Omega} \min_{\eta \in Y_N} \| \psi - \eta \|_{2,\Omega} \\ &\leq \| e \|_{2,\Omega}\, Cp^{-k-2} \| \psi \|_{k+4,\Omega} \\ &\leq Cp^{-k-2} \| e \|_{2,\Omega} \quad \| v \|_{k,\Omega} \\ &\leq Cp^{-k-2}\Re(p,t,\rho(\varphi)) \| v \|_{k,\Omega} \end{aligned}$$

using $\rho(\psi) = (0,0,4+k)$ and the regularity assumption on φ.

□

Remark 3.2 It is not trivial that (3.2) can be solved with the indicated, regular shift theorem holding. It is most easily seen from a construction found in Serbin, [24] or [7], and utilizes the geometry in an essential way. Let ϱ_i denote the reflection about Γ_i, the four straight line segments forming $\partial\Omega$, $i = 1,\cdots,4$. Let $\Omega_i = \varrho_i\Omega, i = 1,\cdots,4$. Now let $\tilde{\psi}\,|_{\Omega_i}(\mathrm{x}) = -\psi(\varrho \mathrm{x})$. Since $\varrho\mathrm{x} \in \Omega$ and $\psi \in C^\infty(\Omega), \tilde{\psi} \in C^\infty(\Omega \cup \bigcup_{i=1}^4 \Omega_i)$. Let

$\tilde{\Omega} = (-3,3)^2$ and define $\tilde{\psi}$ as the odd extension, once more, into the 4 corner regions $(\pm 1, \pm 3)^2$. Now $\tilde{\psi} \in C^\infty(\tilde{\Omega})$ and since Δ^2 is even, $\tilde{\psi}$ satisfies (3.2) on $\tilde{\Omega}$ with $\tilde{v}$ the function defined through the same odd extension procedure. Now standard interior estimates apply, $\Omega \subset\subset \tilde{\Omega}$ and , easily, $\exists C, \| v \|_{s-4,\Omega} = C \| v \|_{s-4,\tilde{\Omega}}; \| \psi \|_{s,\Omega} \leq \| \psi \|_{s,\tilde{\Omega}}$.

$\square$

Remark 3.3 The above remark and thus proposition hold true for a rightangled triangle: $\Omega = \{(x,y) \in R^2 : 0 < x < 1 \quad \wedge \mid y \mid < 1\}$. Now we use that Δ^2 is not only even but also symmetric (in x and y). (When extending ψ as an odd function about the cathetae, $\tilde{\psi}_{xx}(\mathrm{x}) = \psi_{yy}(\varrho \mathrm{x})$, e.g.). The same conclusion applies to the Laplacian, as is well known for the square.

$\square$

Remark 3.4 In general, such a shift theorem will hold when all angles are sufficiently small, [15]. In [12] Chapter 7, it is shown for Δ^2 that each corner of interior angle α supports singular functions with strengths commensurate with the poles τ of the following transcendental characteristic equation

$$\sinh^2(\tau\alpha) = \tau^2 \sin^2 \alpha.$$

See also [18] and [20]. The roots of this equation were tabulated in [9]. It has been shown that for $\alpha \in (0, \pi)$, the first root has imaginary value (for this particular setting the factor determining the strength of the singularity)

$$\Im m \, \tau > \frac{\pi}{\alpha}$$

Therefore the first singular functions belongs to H^s when

$$\alpha < \frac{\pi}{s-2}.$$

We list the bounds for two cases of special interest. For $k = 0$ in (3.1), $s = 4$ and it is sufficient that $\alpha \in (0, \pi/2]$. For $k = 1$ in (3.1), $s = 5$ and $\alpha \in (0, \pi/3]$ suffices.

$\square$

Remark 3.5 Concludingly, the shift theorem for (3.2) and hence the estimate (3.1) will hold for domains with smooth boundary, rectangles, triangles with two sides equal and, with a given ceiling on the shift's upper Sobolev index, for polygonal domains with sufficiently small angles.

Corollary 3.6 *If φ is singular on $\partial\Omega$ with $q = 0$ in (2.7), i.e. $\rho(\varphi) = (\gamma, 0, 1 + \gamma - \epsilon)$, $\epsilon > 0$, and $k \in \mathbf{N}$,*

$$\| e \|_{-k,\Omega} \leq C p^{-k-2(\gamma-1)}. \tag{3.3}$$

4. Interior estimates

We will first derive such estimates for unions of elements within Ω. By a union of elements we mean a single element or a patch of elements. Let $K_0 \subset\subset K_1$ be unions of elements of the triangulation or lattice such that K_i is either a rectangle, a triangle with two sides of equal length, or a polygon with sufficiently small angles. We proceed using the duality ideas of [19]. We shall for simplicity take a most typical case, $\mathbf{F} = \mathbf{0}$. Also, we will assume $\rho(\varphi) = (\gamma, 0, 1 + \gamma - \epsilon)$, $\forall \epsilon > 0$. Obvious modifications can be performed for other cases.

Lemma 4.1 *Let $s \geq 0$ be an integer. Then*

$$\| e \|_{-s,K_0} \leq C(p^{-(s+2)} \| e \|_{2,K_1} + \| e \|_{-s-1,K_1}) \tag{4.1}$$

where C depends on s, K_0, K_1.

Proof: Let B be a ball separating K_0 from K_1, i.e. $K_0 \subset\subset B \subset\subset K_1$ and let $\omega \in C_0^\infty(B)$ with $\omega \equiv 1$ on K_0 (B need only have C^∞ boundary). Then, for $s \geq 0$, we have

$$\| e \|_{-s,K_0} \leq \| \omega e \|_{-1,K_1} = \sup_{f \in H_0^s(K_1) \setminus \{0\}} \frac{(\omega e, f)}{\| f \|_{s,K_1}}.$$

But $\forall f \in H^s(K_1)$, $\exists! v \in H^{s+4}(K_1) \cap H_0^2(K_1)$ such that $c_1(\eta, v) = (\eta, f), \forall \eta \in H_0^2(K_1)$ with $\| v \|_{s+4,K_1} \leq c \| f \|_{s,K_1}$. Here $c_1(\cdot,\cdot)$ denotes $c|_{K_1}(\cdot,\cdot)$. Thus

$$\| e \|_{-s,K_0} \leq C \sup_{v \in H^{s+4}(K_1) \setminus \{0\}} \frac{c_1(\omega e, v)}{\| v \|_{s+4,K_1}}$$

and

$$c_1(\omega e, v) = \int_{K_1} \Delta(\omega e) \Delta v dx = c_1(e, \omega v) + R(e, \omega, v)$$

where

$$\begin{aligned}
| R(e,\omega,v) | &= | \int_{B_1} (e\Delta\omega\Delta v + 2\nabla\omega \cdot \nabla e \Delta v - \Delta e \Delta\omega v - 2\Delta e \nabla\omega \cdot \nabla v) dx | \\
&= | \int_{B_1} e[\Delta\omega\Delta v - 2\nabla \cdot (\Delta v \nabla\omega) - \Delta(v\Delta\omega) - 2\Delta(\nabla\omega \cdot \nabla v)] dx | \\
&\leq C(\omega) \| e \|_{-s-1,K_1} \| v \|_{s+4,K_1}
\end{aligned}$$

using Green's formulae. Also, we use the fact that for some $\psi \in Y_N(K_1)$,

$$\begin{aligned} | c_1(e, \omega v) | &= | c_1(e, \omega v - \psi) | \\ &\leq C \| e \|_{2,K_1} \| \omega v - \psi \|_{2,K_1} \\ &\leq C(\omega) p^{-(s+2)} \| e \|_{2,K_1} \quad \| v \|_{s+4,K_1} \end{aligned}$$

using that K_1 is a union of elements. Otherwise we would get $\| e \|_{2,\Omega}$. This concludes the estimate (4.1).

□

Lemma 4.2 *Let $\varphi \in H^2(\Omega), \varphi_N \in Y_N$ with $p \geq 2$ and let $k \in N$. Then*

$$\| e \|_{0,K_0} \leq C(p^{-2} \| e \|_{2,K_1} + \| e \|_{-k,K_1}) \tag{4.2}$$

where C depends on K_0, K_1, k.

Proof: For $j \in Z_+$, denote the index $i(j) = \frac{j}{j+1}$. Let $K_0 \subset\subset K_{1/2} \subset\subset \cdots \subset\subset K_{i(k)} = K_1$ be unions of elements. From the previous lemma with $s = 0$, we get

$$\| e \|_{0,K_0} \leq C(p^{-2} \| e \|_{2,K_{1/2}} + \| e \|_{-1,K_{1/2}})$$

Successively applying the previous lemma to estimate $\| e \|_{-j,K_{i(j)}}$, we get the estimate.

□

An easy consequence of (4.2) is obtained by interpolation between (4.2) and the obvious $\|e\|_{2,K_0} \leq C(\|e\|_{2,K_1} + \|e\|_{-k,K_1})$ to get

$$\|e\|_{1,K_0} \leq C(p^{-1}\|e\|_{1,K_1} + \|e\|_{-k,K_1}) \tag{4.2$'$}$$

The next lemma provides a local growth estimate on the H^2-norm of φ_N.

Lemma 4.3 *Let $k \in N$. Then*

$$\| \varphi_N \|_{2,K_0} \leq C(p^{-\min\{1,2(\gamma-1)\}} \| \varphi_N \|_{2,K_1} + \| \varphi_N \|_{-k,K_1}) \tag{4.3}$$

where C depends on K_0, K_1, k and φ.

Proof: Let π be the elliptic projection of $H_0^2(K_1)$ onto $Y_N(K_1)$ defined by:

$$c_1(v - \pi v, \psi) = 0, \quad \forall \psi \in Y_N(K_1).$$

π satisfies

$$\begin{aligned}
\| \pi v \|_{2,K_1} &\leq C\sqrt{c_1(\pi v, \pi v)} \\
&\leq C \sup_{\psi \in Y_N(B_1)\setminus\{0\}} \frac{c_1(\pi v, \psi)}{\| \psi \|_{2,K_1}} \\
&\leq C \sup_{\psi \in Y_N(K_1)\setminus\{0\}} \frac{c_1(v, \psi)}{\| \psi \|_{2,K_1}} \\
&\leq C \| v \|_{2,K_1} .
\end{aligned}$$

Let B be a ball separating K_0 from K_1, i.e. $K_0 \subset\subset B \subset\subset K_1, \omega \in C_0^\infty(B)$ with $\omega \equiv 1$ on K_0. Then

$$\| \varphi_N \|_{2,K_0} \leq \| \omega\varphi_N \|_{2,K_1} \leq \| \omega\varphi_N - \pi(\omega\varphi_N) \|_{2,K_1} + \| \pi(\omega\varphi_N) \|_{2,K_1}$$

Let $\rho(\varphi) = (\gamma, 0, 1+\gamma-\epsilon)$ so $\varphi \in H^{1+\gamma-\epsilon}(\Omega)$, for $\gamma > 1$. Then $\omega\varphi \in H^{1+\gamma-\epsilon}(\Omega)$ and is zero in a neighbourhood of ∂K_1, so there exists $\psi \in Y_N(K_1)$ so that

$$\| \omega\varphi - \psi \|_2 \leq C(\omega, \varphi) p^{-2(\gamma-1)} \| \varphi \|_2$$

Also $\| \varphi - \varphi_N \|_2 \leq C(\varphi) p^{-2(\gamma-1)} \| \varphi \|_2$. Finally, $\| \varphi \|_2 \leq \| \varphi - \varphi_N \|_2 + \| \varphi_N \|_2 \leq C(\varphi) \| \varphi_N \|_2$, so that

$$\begin{aligned}
\| \omega\varphi_N - \pi(\omega\varphi_N) \|_{2,K_1} &\leq C \| \omega\varphi_N - \psi \|_{2,K_1} \\
&\leq C[\|\omega(\varphi - \varphi_N)\|_{2,K_1} + \|\omega\varphi - \psi\|_{2,K_1}] \\
&\leq C(\omega, \varphi) p^{-2(\gamma-1)} \| \varphi_N \|_{2,K_1} .
\end{aligned}$$

The other addend

$$\| \pi(\omega\varphi_N) \|_{2,K_1} \leq C c_1(\pi(\omega\varphi_N), \overline{\varphi}) = C c_1(\omega\varphi_N, \overline{\varphi})$$

where $\overline{\varphi} = \pi(\omega\varphi_N) / \| \pi(\omega\varphi_N) \|_{2,K_1}$ so that $\| \overline{\varphi} \|_{2,K_1} = 1$.

$$c_1(\omega\varphi_N, \overline{\varphi}) = c_1(\varphi_N, \omega\overline{\varphi}) + R(\varphi_N, \omega, \overline{\varphi})$$

as before and we get that

$$| c_1(\omega\varphi_N, \overline{\varphi}) | \leq C(p^{-2(\gamma-1)} \| \varphi_N \|_{2,K_1} + \| \varphi_N \|_{1,B})$$

since $c_1(\varphi_N, \psi) = 0$ for $\mathbf{F} = \mathbf{0}$ and supp $\omega \subset B$. Recalling (4.2) with $e = \varphi_N$ and its corollary (4.2′),

$$\| \pi(\omega\varphi_N) \|_{2,K_1} \leq C(p^{-\min\{1, 2(\gamma-1)\}} \| \varphi_N \|_{2,K_1} + \| \varphi_N \|_{-k,K_1}).$$

Adding the two upper bounds yields (4.3).

□

Lemma 4.4 *Under the hypothesis of the previous lemma, with* $0 \le k \le 5$

$$\| \varphi_N \|_{2,K_0} \le C \| \varphi_N \|_{-k,K_1} \tag{4.4}$$

with C *depending on* K_0, K_1, k *and* $\rho_1(\varphi) = \gamma \ge 2 - \frac{3}{k+1}$.

Proof: Let $K_0 \subset\subset K_{1/2} \subset\subset \cdots \subset\subset K_{i(k+1)} \subset\subset K_1$ be unions of elements; i was defined just below (4.2). Lemma 4.3 applies to each pair $K_{i(j)}, K_{i(j+1)}, 0 \le j \le k$. So for $\gamma \le 3/2$,

$$\| \varphi_N \|_{2,K_{i(j)}} \le C(p^{-2(\gamma-1)} \| \varphi_N \|_{2,K_{i(j+1)}} + \| \varphi_N \|_{-k,K_{i(j+1)}})$$

Iterating from $j = 0$ to $k+1$, we get

$$\| \varphi_N \|_{2,K_0} \le C(p^{-2(\gamma-1)(k+1)} \| \varphi_N \|_{2,K_{i(k+1)}} + \| \varphi_N \|_{-k,K_{i(k+1)}})$$

Let $K_{i(k+1)} \subset\subset K' \subset\subset K_1$. But, by Schmidt's inequality [10],

$$\begin{aligned} p^{-2(k-2)} \| \varphi_N \|_{2,K_{i(k+1)}} &\le p^{-2(k-2)} \| \varphi_N \|_{2,K'} \\ &\le C \| \varphi_N \|_{-k,K'} \\ &\le C \| \varphi_N \|_{-k,K_1}, \end{aligned}$$

since $(k+1)(\gamma-1) \ge k-2$ iff $\gamma \ge 2 - \frac{3}{k+1}$ and this last expression $\le 3/2$ iff $k \le 5$. The case $\gamma > 3/2$ follows in a similar way, again we need $k \le 5$.

□

Next the local version of the main result of the section.

Lemma 4.5 *Let* $K_0 \subset\subset K_1$ *be unions of elements,* $\varphi \in H^t(K_1), \varphi_N \in Y_N$ *and* $k \in \mathrm{N}$. *Then for* $2 \le s < 1 + \gamma$,

$$\| e \|_{s,K_0} \le C(p^{-2(\gamma-s+1)} + p^{2(s-2)} \| e \|_{-k,K_1}) \tag{4.5}$$

Proof: Let $K_0 \subset\subset K_{1/3} \subset\subset K_{2/3} \subset\subset K_1$ and $\omega \equiv 1$ on $K_{1/3}, \omega \in C_0^\infty(K_{2/3})$. Then

$$\begin{aligned} \| \omega\varphi - \pi(\omega\varphi) \|_{2,K_1} &\le C \inf_{\psi \in Y_N(K_1)} \| \omega\varphi - \psi \|_{2,K_1} \\ &\le Cp^{-2(\gamma-1)} \end{aligned}$$

and

$$\| \varphi - \varphi_N \|_{2,K_0} \le \| \varphi - \pi(\omega\varphi) \|_{2,K_0} + \| \pi(\omega\varphi) - \varphi_N \|_{2,K_0}$$

where $c_1(\varphi_N - \pi(\omega\varphi), \psi) = 0$, $\forall \psi \in Y_N(K_{1/3})$ so that lemma 4.4 applies with $K_{1/3}$ replacing K_1,

$$\begin{aligned} \| \varphi_N - \pi(\omega\varphi) \|_{2,K_0} &\le C \| \varphi_N - \pi(\omega\varphi) \|_{-k,K_{1/3}} \\ &\le C(\| e \|_{-k,K_{1/3}} + \| \omega\varphi - \pi(\omega\varphi) \|_{-k,K_{1/3}}) \\ &\le C(\| e \|_{-k,K_1} + \| \omega\varphi - \pi(\omega\varphi) \|_{-k,K_1}) \end{aligned}$$

and collecting, we get

$$\begin{aligned} \| e \|_{2,K_0} &= \| \varphi - \varphi_N \|_{2,K_0} \\ &\le C(\| \omega\varphi - \pi(\omega\varphi) \|_{-k,K_1} + \| e \|_{-k,K_1}) \\ &\le C(p^{-2(\gamma-1)} + \| e \|_{-k,K_1}) \end{aligned}$$

which is (4.5) when $s = 2$. Let $s \ge 2, K_0 \subset\subset K_{1/3} \subset\subset K_{2/3} \subset\subset K_1$ as before. Then for some $\psi \in Y_N$,

$$\begin{aligned} \| e \|_{s,K_0} &\le \| \varphi - \psi \|_{s,K_1} + \| \psi - \varphi_N \|_{s,K_{2/3}} \\ &\le \| \varphi - \psi \|_{s,K_1} + p^{2(s-2)} \| \psi - \varphi \|_{2,K_1} + p^{2(s-2)} \| e \|_{2,K_1} \\ &\le C(p^{-2(\gamma-s+1)} + p^{2(s-2)-2(\gamma-s+1)} + p^{2(s-2)} \| e \|_{-k,K_1}) \\ &\le C(p^{-2(\gamma-s+1)} + p^{2(s-2)} \| e \|_{-k,K_1}) \end{aligned}$$

which is (4.5).

□

The main result generalizes (4.5) to hold for $\Omega_0 \subset\subset \Omega_1$.

Theorem 4.6 *Let $\Omega_0 \subset\subset \Omega_1 \subset\subset \mathbf{R}^2$, $\varphi \in H^t(\Omega_1)$, $\varphi_N \in Y_N$ and $k \in \mathbf{N}$. Then for $2 \le s < 1 + \gamma$,*

$$\| e \|_{s,\Omega_0} \le C(p^{-2(\gamma-s+1)} + p^{2(s-2)} \| e \|_{-k,\Omega_1}). \tag{4.6}$$

We may think of $\Omega_1 = \Omega$ or $\Omega_1 \subset\subset \Omega$ depending on the regularity of φ.

Proof: Cover $\overline{\Omega_0}$ with a finite number of $K_0(\mathbf{x}_i)$, $i = 1, \ldots, m$ centered at $\mathbf{x}_i \in \Omega_0$ with sidelengths $= \frac{1}{2}\mathrm{dist}(\overline{\Omega_0}, \partial\Omega_1)$. Let $K_1(\mathbf{x}_i)$ be the same except with sidelengths $= \mathrm{dist}(\overline{\Omega_0}, \partial\Omega_1)$. For $s \ge 0$, we have from the previous lemma that

$$\begin{aligned} \| e \|_{s,K_0(\mathbf{x}_i)} &\le C_i(p^{-2(\gamma-s+1)} + p^{2(s-2)} \| e \|_{-k,K_1(\mathbf{x}_i)}) \\ &\le C_i(p^{-2(\gamma-s+1)} + p^{2(s-2)} \| e \|_{-k,\Omega_1}) \end{aligned}$$

which yields (4.6).

□

Corollary 4.7 *With the same hypotheses as in Thm. 4.6,*

$$\| e \|_{s,\Omega_0} \leq Cp^{-2(\gamma-s+1)}. \tag{4.7}$$

Proof: For $k \geq 0$, $2(s-2)+k-2\gamma > -2(\gamma-s+1)$ and applying cor. 3.6 yields (4.7).

□

To obtain (4.7) it is thus sufficient to use $k=0$, i.e. no restrictions are needed on γ other than $\gamma > 0$, see lemma 4.4. This provides quasi-optimal error estimates for $s > 2$.

Remark 4.8 For $\mathbf{F} \neq 0$, we can modify the lemmas to reflect φ_N satisfying instead

$$E(\psi) \stackrel{\text{def}}{=} c_1(e,\psi) = c_1(\varphi - \varphi_N, \psi),\ \forall \psi \in Y_N(K_1) \tag{4.8}$$

where $E \in [H_0^2(K_1)]^*$. For $s \geq -2$, $s \in \mathbf{Z}$, let $\| E \|_{-s,\Omega} = \sup_{v \in H_0^{s+2}(\Omega)} |E(\psi)| / \| v \|_{s+2,\Omega}$. If φ_N satisfies (4.8), lemma 4.1 should state instead,

$$\|e\|_{-s,K_0} \leq C(p^{-4}\|e\|_{2,K_1} + \|e\|_{-s-1,K_1} + p^{-4}\|E\|_{2,K_1} + \|E\|_{-s,K_1});$$

and lemma 4.2,

$$\|e\|_{0,K_0} \leq C(p^{-4}\|e\|_{2,K_1} + \|e\|_{-k,K_1} + p^{-4}\|E\|_{2,K_1} + \|E\|_{0,K_1}).$$

The remaining lemmas and theorems hold unchanged. In the proofs one has to carry along and estimate terms $\|E\|_r$.

□

Remark 4.9 It is not always necessary to have $\Omega_0 \subset\subset \Omega_1$ or $K_0 \subset\subset K_1$, since it is at times possible to extend about parts of the boundary by the methods mentioned in remarks 3.2 and 3.3 (or the smoothness of the solution otherwise warrants this). In such cases, we may use the estimates up to that part of the boundary.

□

5. A projection method.

We now discuss how to recover the hydrostatic pressure utilizing that it satisfies the Poisson problem (1.6). We shall see that there seems to be a fundamental interplay between the lacks of stability and regularity.

First suppose P is smooth, i.e. $P \in H^s \cap W$ for some $s > 1$. We will as before assume that the regularity of P is dictated by that near the boundary of Ω. We will then use the following weak formulation

$$\text{Find } P \in H^1 \cap W \text{ such that } \forall q \in H^1(\Omega) \cap W,$$

$$d_1(P,q) = g(\mathbf{U},q) + (\mathbf{F},\nabla q) \tag{5.1}$$

where the forms in (5.1) are defined by

$$d_1(P,q) = \int_\Omega \nabla P \cdot \nabla q \, d\mathbf{x},$$

$$g(\mathbf{U},q) = -\int_{\partial\Omega} \Delta\mathbf{U} \cdot \mathbf{n} \, q \, ds. \tag{5.2}$$

In order to discretize, we use instead $g(\mathbf{U}_N, q)$, which is, of course, why we needed higher order error estimates for $\|\mathbf{U} - \mathbf{U}_N\|$, since for $s \geq 0$,

$$\|\Delta\mathbf{U} \cdot \mathbf{n} - \Delta\mathbf{U}_N \cdot \mathbf{n}\|_{s-3/2,\partial\Omega} \leq C\|\mathbf{U} - \mathbf{U}_N\|_{1+s,\Omega}. \tag{5.3}$$

Note that it is possible (and sometimes desirable) to pose (5.1) locally, i.e. over a union of elements, rather than over all of Ω. We may discretize selecting $W_N = P^{p-1} \cap W$. We will estimate in turn each of the two terms on the right-hand-side of

$$\|P - P_N\|_{1,K_0} \leq \|P - P_{\mathbf{U}_N}\|_{1,K_0} + \|P_{\mathbf{U}_N} - P_N\|_{1,K_0}$$

where $P_{\mathbf{U}_N}$ satisfies $d_1(P_{\mathbf{U}_N}, q) = g(\mathbf{U}_N, q)$, $\forall q \in W_N$. Let $q = P - P_{\mathbf{U}_N}$, then since d_1 is symmetric and coerces the 1-norm,

$$\begin{aligned} \|P - P_{\mathbf{U}_N}\|_{1,\Omega}^2 &\leq \|P - P_{\mathbf{U}_N}\|_{3/2-s,\partial\Omega} \|\Delta(\mathbf{U} - \mathbf{U}_N) \cdot \mathbf{n}\|_{s-3/2,\partial\Omega} \\ &\leq C\|P - P_{\mathbf{U}_N}\|_{2-s,\Omega} \|\mathbf{U} - \mathbf{U}_N\|_{1+s,\Omega} \end{aligned}$$

which with $s = 1$ and (4.7) gives

$$\|P - P_{\mathbf{U}_N}\|_1 \leq Cp^{-2(\gamma-2)}$$

For the second addend, P inherits a singularity from $\mathbf{U}$ in (1.6), $\tilde{P} = r^{\gamma-2}\phi(\theta)$ so that at worst,

$$\|P_N - P_{\mathbf{U}_N}\|_1 \leq Cp^{-2(\gamma-2)}$$

Hence,

$$\|P - P_N\|_1 \leq Cp^{-2(\gamma-2)}. \tag{5.4}$$

Example 5.1 Consider the "driven cavity" flow problem

$$\begin{aligned} -\Delta\mathbf{U} + \nabla P &= 0 \text{ in } R = (-1,1)^2 \\ \nabla \cdot \mathbf{U} &= 0 \text{ in } R \\ \mathbf{U}(x,1) &= (1-x^2)^\beta \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ for } -1 < x < 1 \\ \mathbf{U} &= 0 \text{ elsewhere on } \partial R. \end{aligned}$$

where $\beta > 0$. According to Thm. 6.2 of [2], the last three equations has a solution in $[H^{1+\beta-\epsilon}(R)]^2$ and no solution is in $[H^{1+\beta}(R)]^2$. This then carries over to hold for the solution of the full system by known regularity results for Stokes problem [18], [12].

$\square$

Proposition 5.2 *Let* $\Omega = (-1,1)^2$. $W_N = P_{p-1} \cap W$ *and*

$$\begin{array}{c} \textit{Find } P_N \in W_N \textit{ such that } \forall q \in W_N, \\ d_1(P_N, q) = g(\mathbf{U}_N, q) \end{array} \tag{5.5}$$

where d_1, g *and* $\mathbf{U}_N \in (H_0^1)^2$ *have been defined above. For the driven cavity,* $\beta > 1$, *we obtain*

$$\begin{aligned} \|P - P_N\|_{1,\Omega} &\leq Cp^{-2(\gamma-2)} \textit{ and} \\ \|P - P_N\|_{0,\Omega} &\leq Cp^{-2(\gamma-1)} \end{aligned} \tag{5.6}$$

Note that the second half of (5.6) corresponds exactly to the rate of approximation of the velocity in H^1.

Proof: For the square, the odd extension procedure works for $\mathbf{U}, \mathbf{U}_N \in (H_0^1)^2$ and $\mathbf{U}|_{(-1,1)\times\{1\}} \in H_0^{2\beta}((-1,1) \times \{1\})$ in such a way that the smoothness properties are preserved and the norms of the extended functions remain the same as those of the original functions. Applying (5.4) yields the 1-norm estimate in (5.6). To get the second half, we employ Aubin-Nitsche's construction: Let $\delta = P - P_N$ and $\chi \in H_0^1(\Omega)$ solve the auxilliary problem

$$d_1(\chi, \psi) = (\delta, \psi), \quad \forall \psi \in W.$$

Then $\|\chi\|_2 \leq C\|\delta\|_0$ because Ω is convex. $\|\delta\|_0^2 = (\delta, \delta) = d_1(\delta, \chi) = d_1(\delta, \chi - \psi) \leq C\|\delta\|_1\|\chi - \psi\|_1 \leq Cp^{-2}\|\delta\|_1\|\delta\|_0$ as we can choose ψ so that $\|\chi - \psi\|_1 \leq Cp^{-2}\|\chi\|_2$.

$\square$

Remark 5.3 Thus we are able to explain why it should be possible to get optimal convergence rate also for the pressure as observed in [16] for $\beta = 2$. P_N defined in (5.5) solves the same problem as in the method from [16] since we observe from Prop. 2.1 that $\exists p_0 \in \mathbf{Z}_+$, such that for $p > p_0$, $b(\mathbf{U}_N, q) = 0$, $\forall q \in W_N$ and $b(\mathbf{v}, P_N) = 0$, $\forall \mathbf{v} \in Z_N$, rotations of streamfunctions. Now let $\mathbf{v} = \nabla q$ in (1.3) and integrate by parts to get (5.5).

$\square$

Remark 5.4 With P smooth, it is also possible to introduce mixed methods to solve for P_N as in [26] and use either Raviart-Thomas or Brezzi-Douglas-Marini elements to get almost optimal order convergence for ∇P.

□

Remark 5.5 We can interpret (5.1) as local averaging bringing this method into the same general frame of ideas as that of [7] or [17].

□

For a pressure P which is not assumed smooth globally, $P \in H^s \cap W$ for some $s > 0$, the above procedure can be employed locally on an element or a patch of elements in the interior of Ω, where P could be assumed smooth. This would yield an interior optimal rate approximation to P.

For P still not smooth one might be tempted to modify (5.1) to some other weak form of (1.6) presupposing minimal smoothness on P.

Example 5.6 In stead of (5.1) take the following problem on $\Omega = (-1,1)^2$:

$$\begin{array}{c} \text{Find } P \in W \text{ such that } \forall q \in H^2 \cap \{q : \frac{\partial q}{\partial n} = 0 \text{ on } \partial\Omega\} \\ d_2(P,q) = g(\mathbf{U},q) \end{array} \tag{5.7}$$

where the bilinear form d_2 is defined by

$$d_2(P,q) = -\int_\Omega P \Delta q \, dx \tag{5.8}$$

Note that for the discrete form of (5.7), P_N would be identical to that of (5.5) since $0 = \int_{\partial\Omega} P_N \frac{\partial q}{\partial n} \, ds = d_2(P_N,q) + d_1(P_N,q)$ for $q \in W_N$. So one might expect to get optimal rate 0-norm estimates for $P - P_N$, since also

$$\inf_{P \in W} \sup_{q \in H^2, \, \frac{\partial q}{\partial n} = 0 \text{ on } \partial\Omega} \frac{d_2(P,q)}{\|P\|_0 \|q\|_2} \geq c > 0$$

because $\forall p \in W \setminus \{0\}, \exists q^* \in H^2 \cap \{\frac{\partial q}{\partial n} = 0 \text{ on } \partial\Omega\}$ such that $\Delta q^* = p$, $\|q^*\|_2 \leq C\|p\|_0$. Nevertheless one will fail to get optimal rate of convergence since d_2 is p-unstable, see (2.2.28) in [3] for one dimension.

□

6. A method using potentials

We will demonstrate pressure recovery, still using the Poisson equation (1.6), but now in a way so that we get the hydrostatic pressure evaluated at an interior point of Ω. Let $\mathbf{F} = 0$ so that P satisfies, for any $\Omega_1 \subset \Omega$ with $\partial\Omega_1$ sufficiently regular,

$$\begin{aligned} \Delta P &= 0 \text{ in } \Omega_1 \\ \frac{\partial P}{\partial n} &= (\Delta \mathbf{U}) \cdot \mathbf{n} \text{ on } \partial\Omega_1 \end{aligned} \tag{6.1}$$

Then , if $\partial\Omega_1 \in C^3$,

$$P(\mathbf{x}) = -\frac{1}{|\partial\Omega_1|}\int_{\partial\Omega_1} \log\frac{1}{|\mathbf{x}-\mathbf{y}|}(\Delta\mathbf{U}\cdot\mathbf{n})(\mathbf{y})\, ds(\mathbf{y}) \tag{6.2}$$

and $P_{\mathbf{U}_N}$ is obtained by replacing $\mathbf{U}$ in (6.2) by $\mathbf{U}_N$.

Corollary 6.1 *If $P_{\mathbf{U}_N}$ is evaluated exactly on $\partial\Omega_1$, smooth, $x \in \Omega_0 \subset\subset \Omega_1 \subset\subset \Omega$ according to (6.2) with $\mathbf{U}_N$ in stead of $\mathbf{U}$, and $\mathbf{U} \in H^{1+s}(\Omega_1)$ for some $s > 0$ and of singularity strength $\gamma - 1$, then for $k \in \mathbf{N}$,*

$$|P - P_{\mathbf{U}_N}|(\mathbf{x}) \leq C(\Omega_1)p^{-k-2(\gamma-1)} \tag{6.3}$$

Note this represents a superconvergence result for the p version in the interior of elements.

Proof: From (5.3), (3.3), and the fact that the single layer potential is smooth away from $\partial\Omega_1$.

□

Remark 6.2 It might be possible to deal with $\partial\Omega_1 \to \partial\Omega$ by using explicit knowledge about $\mathbf{U}$ near corners of Ω. Such explicit knowledge can be generated a priori, e.g. using the methods of P. Papadakis, [21].

□

Remark 6.3 It might be desirable to evaluate the right hand side of (6.2) with $\mathbf{U}_N$ replacing $\mathbf{U}$ by quadrature.

□

Remark 6.4 We note that this procedure can be performed without smoothness assumptions other that $\mathbf{U} \in H^{1+s}(\Omega_1)$, i.e. for $P \in H^s$, $s > 0$.

□

Other such extraction procedures have been investigated for nearly incompressible plane-strain elasticity in [27]. The two continuous problems are connected as shown in [29], Chap. 6.

7. Concluding remarks

We have introduced two methods for recovering the hydrostatic pressure in Stokes problem. The necessity for this was demonstrated in [16]. We derived negative norm and interior estimates enabling us to prove optimal convergence rate in the interior elements and for special geometries and boundary conditions up to the boundary. Thus we were able to explain the ability of the mixed method introduced in [16] to approximate P at optimal rates when the boundary data were sufficiently smooth.

References

[1] R. A. Adams. *Sobolev Spaces*. Academic Press, 1975.

[2] D. Arnold, L. R. Scott, and M. Vogelius. Regular inversion of the divergence operator with Dirichlet conditions on a polygon. *Ann. Sc. Norm. Super. Pisa, Cl. Sci., IV Ser.*, 15(2):169–192, 1988.

[3] D. N. Arnold, I. Babuška, and J. Osborn. Finite element methods: principles for their selection. *Comput. Methods Appl. Mech. Engrg.*, 45:57–96, 1984.

[4] I. Babuška. Error bounds for the finite element method. *Numer. Math.*, 16:322–333, 1971.

[5] I. Babuška and M. Suri. On the optimal convergence rate of the p-version of the finite element method. *SIAM J. Numer. Anal.*, 24:750–776, 1987.

[6] I. Babuška, B. Szabó, and I. N. Katz. The p version of the finite element method. *SIAM J. Numer. Anal.*, 18:515–545, 1981.

[7] J. H. Bramble and A. H. Schatz. Higher order local accuracy by averaging in the finite element method. *Math. Comput.*, 31(137):94–111, 1977.

[8] F. Brezzi. On the existence, uniqueness and approximation of saddlepoint problems arising from Lagrangian multipliers. *RAIRO*, 8:129–151, 1974.

[9] M. Dauge. Stationary Stokes and Navier-Stokes systems on two- or three-dimensional domains with corners. part I: Linearized equations. *SIAM J. Math. Anal.*, 20(1):74–97, 1989.

[10] M. Dorr. The approximation theory for the p-version of the finite element method. *SIAM J. Numer. Anal.*, 21(6):1180–1207, 1984.

[11] D. Gottlieb and S. Orszag. *Numerical Analysis of Spectral Methods: Theory and Applications*. Volume 26 of *CBMS-NSF Regional Conf. Series in Applied Math.*, SIAM, 1977.

[12] P. Grisvard. *Elliptic Problems in Nonsmooth Domains*. Pitman, 1985.

[13] T. J. R. Hughes, L. P. Franca, and M. Balestra. A new finite element formulation for computational fluid dynamics: V. Circumventing the Babuska-Brezzi condition: A stable Petrov-Galerkin formulation of the Stokes problem accomodating equal-order interpolation. *Comput. Methods Appl. Mech. Eng.*, 59:85–99, 1986.

[14] J. Douglas, Jr. and J. Wang. An absolutely stabilized finite element method for the Stokes problem. *Math. Comput.*, 52(186):495–508, 1989.

[15] S. Jensen. An H_0^m interpolation result. Submitted to SIAM J. Math. Anal.

[16] S. Jensen and M. Vogelius. Divergence stability in connection with the p version of the finite element method. *RAIRO, Modelisation Math. Anal. Numer.*, 1990. To appear.

[17] C. Johnson and J. Pitkaranta. Analysis of some mixed finite element methods related to reduced integration. *Math. Comput.*, 38(158):357–400, 1982.

[18] R. B. Kellogg and J. E. Osborn. A regularity result for the Stokes problem in a convex polygon. *J. Funct. Anal.*, 21(4):397–431, 1976.

[19] J. A. Nitsche and A. H. Schatz. Interior estimates for Ritz-Galerkin methods. *Math. Comput.*, 28(128):937–958, 1974.

[20] J. Osborn. Regularity of solutions of the Stokes problem in a polygonal domain. In B. Hubbard, editor, *Symposium on Numerical Solutions of Partial Differential Equations III*, pages 393–411, 1975.

[21] P. Papadakis. *Computational aspects of the determination of the stress intensity factors for two-dimensional elasticity.* PhD thesis, Univ. of Maryland, 1988.

[22] E. M. Rønquist. *Optimal spectral element method for unsteady three-dimensional incompressible Navier-Stokes equations.* PhD thesis, MIT, 1988.

[23] L. R. Scott and M. Vogelius. Conforming finite element methods for incompressible and nearly incompressible continua. In *Lectures in Applied Mathematics*, pages 221–244, AMS, 1985.

[24] S. Serbin. *A Computational Investigation of Least Squares and other Projection Methods for the Approximate Solution of Boundary Value Problems.* PhD thesis, Cornell Univ., 1971.

[25] M. Suri. The *hp* version of the finite element method for elliptic problems of order 2 *l*. *RAIRO, Modelisation Math. Anal. Numer.*, 24:107–146, 1990.

[26] M. Suri. On the stability and convergence of higher order mixed finite element methods for second order elliptic problems. *Math. Comput.*, 54(189):1–19, 1990.

[27] B. Szabó, I. Babuška, and B. K. Chayapathy. Stress computations for nearly incompressible materials by the p-version of the finite element method. *Int. J. Numer. Methods Eng.*, 28(9):2175–2190, 1989.

[28] B. A. Szabó. *PROBE: Theoretical Manual.* Noetic Technologies Corp., St. Louis, Missouri, 1985.

[29] R. Temam. *Navier-Stokes equations.* North-Holland, 1977.

[30] M. Vogelius. A right-inverse for the divergence operator in spaces of piecewise polynomials – Application the the p version of the finite element method. *Numer. Math.*, 41:19–37, 1983.